Stationary Euler flows near the Kolmogorov and Poiseuille flows

Necas PDE seminar - Mathematical Institute of the Czech Academy of Sciences

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In a 2d domain, consider

$$\begin{cases} \partial_t \boldsymbol{U} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{U} + \nabla P = \nu \Delta \boldsymbol{U}, \\ \nabla \cdot \boldsymbol{U} = 0. \end{cases}$$

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- P is the scalar pressure
- $\nu \geq 0$ is the inverse Reynolds number
  - $\nu = 0$: Inviscid fluid $\rightarrow$ Euler equations
  - $\nu > 0$: Viscous fluid $\rightarrow$ Navier-Stokes equations

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In vorticity formulation $\Omega = \nabla^\perp \cdot \boldsymbol{U} = -\partial_y U_1 + \partial_x U_2$:

$$\begin{cases} \partial_t \Omega + \boldsymbol{U} \cdot \nabla \Omega = \nu \Delta \Omega, \\ \boldsymbol{U} = \nabla^\perp \Psi, \quad \Delta \Psi = \Omega. \end{cases}$$

Main features

- Smooth solutions remain smooth and are global ($\nu \geq 0$)
- All L^p norms are conserved ($\nu = 0$)

What happens as $t \to \infty$?

- In (bounded) domains, all mean-zero solutions decay to 0 ($\nu > 0$)
- For $\nu = 0$, the dynamics can be very complicated: there is no global relaxation mechanism

Vorticity mixing

Mixing can be thought of as a cascading process in which information travels to smaller and smaller spatial scales.

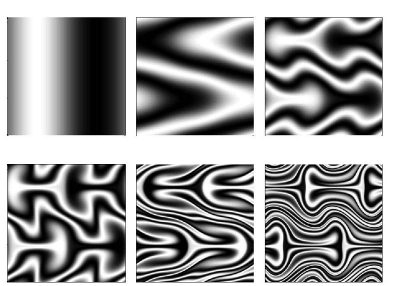

Figure 1: No diffusion (Doering et al.)

Understanding this fundamental process sheds light on:

- Relaxation towards stationary states and coherent structures
- Meta-stable behavior in ocean/atmospheric models
- The derivation of turbulence scaling laws (Kolmogorov, Batchelor)

A conjecture

Longtime behavior for 2D Euler

The generic solution to the 2D Euler equations in vorticity form on $\mathbb{T}^2$ is such that the orbit $\{\Omega(t) : t \in \mathbb{R}\}$ is not precompact in $L^2(\mathbb{T}^2)$.

- All solutions that experience some vorticity mixing as $t \to \infty$ are not precompact (very hard to prove in general!)
- Understand the dynamics near steady states such as shear flows and vortices
- Understand the (local) structure of known steady states

Steady states and perturbations

Consider a given equilibrium $\boldsymbol{U}^S$, write $\boldsymbol{U} = \boldsymbol{U}^S + \boldsymbol{u}$:

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{U}^S \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{U}^S + \nabla p = \nu \Delta \boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \\ \nabla \cdot \boldsymbol{u} = 0, \qquad \boldsymbol{u}(t=0) = \boldsymbol{u}^{in}. \end{cases}$$

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Motto: "Linear stability implies some nonlinear stability"

- Linear spectral / mode stability (e.g. Rayleigh-Fjørtoft)
- Linear Lyapunov stability (but, non-normality and $\nu \to 0 \ldots$)

However...

Dynamics near steady states

Given two norms X and Y, the equilibrium $\boldsymbol{U}^S$ is called (asymptotically) stable with exponent $\gamma \geq 0$ if

$$\|\boldsymbol{u}^{in}\|_X \lesssim \nu^\gamma \qquad \Rightarrow \qquad \begin{cases} \|\boldsymbol{u}(t)\|_Y \ll 1, & \forall t > 0, \\ \|\boldsymbol{u}(t)\|_Y \to 0, & \text{as } t \to \infty. \end{cases}$$

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No canonical choice of norms

- choice of Y: from linear dynamics
- choice of X: not unique, γ can depend on X
- when $\nu = 0$ take $\gamma = 0$, $\forall \epsilon \, \exists \delta$...

Shear flows

Class of equilibria: shears $\boldsymbol{U}^S = (v(y), 0)$.

In vorticity form:

$$\begin{cases} \partial_t \omega + v(y)\partial_x \omega - v''(y)\partial_x \psi - \nu \Delta \omega = -\boldsymbol{u} \cdot \nabla \omega, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \omega. \end{cases}$$

- Couette: $v(y) = y$, on $\mathbb{T} \times \mathbb{R}$
- Poiseuille: $v(y) = y^2$, on $\mathbb{T} \times \mathbb{R}$
- Kolmogorov: $v(y) = \sin y$, on $\mathbb{T}^2$

The Couette flow

Couette flow

If $v(y) = y$:

$$\partial_t \omega + y \partial_x \omega = \nu \Delta \omega.$$

The solution is explicit:

$$\widehat{\omega}(t, k, \eta) = \widehat{\omega}^{in}(k, \eta + kt) \exp\left\{-\nu \int_0^t \left[k^2 + |\eta + kt - k\tau|^2\right] \mathrm{d}\tau\right\}.$$

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- If $\nu = 0$, info goes to high frequencies (when $k \neq 0$).
  Inviscid damping:
  $$\|u_1(t) - \langle u_1(t) \rangle_x\|_{L^2} \lesssim \frac{1}{\langle t \rangle} \|\omega^{in}\|_{H^1},$$
  $$\|u_2(t)\|_{L^2} \lesssim \frac{1}{\langle t \rangle^2} \|\omega^{in}\|_{H^2}.$$

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  $$\|u_2(t)\|_{L^2} \lesssim \frac{1}{\langle t \rangle^2} \|\omega^{in}\|_{H^2}.$$
- If $\nu > 0$, then we have enhanced dissipation:
  $$\|\omega(t) - \langle \omega(t) \rangle_x\|_{L^2}^2 \leq \|\omega^{in} - \langle \omega^{in} \rangle_x\|_{L^2}^2 \mathrm{e}^{-\frac{1}{6}\nu t^3}.$$

Results

What happens at the nonlinear level?

For small perturbations, $\boldsymbol{u}(t, x, y) \to (u^{\infty}(y), 0)$ as $t \to \infty$

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For $\nu = 0$: $\|\omega^{in}\|_X \lesssim \varepsilon$ implies inviscid damping?

- Bedrossian, Masmoudi '13: if the perturbation is small in Gevrey-2^-, then inviscid damping holds. That is, sufficiently smooth, non-shear perturbations of the Couette flow $\boldsymbol{U} = (y,0)$ undergo vorticity mixing and inviscid damping.
- Deng, Masmoudi '18: Gevrey-2 is optimal.

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For $\nu > 0$: $\|\omega^{in}\|_X \lesssim \nu^{\gamma}$ implies enhanced dissipation?

- Bedrossian, Masmoudi, Vicol '14: $\gamma = 0$ if X is Gevrey-2^-.
- Bedrossian, Vicol and Wang '16: $\gamma = 1/2$ if $X = H^s$, $s > 1$.
- Masmoudi, Zhao '19: $\gamma = 1/3$ if $X = H^s$, $s > 40$.

Linear results

- 2D Euler (after 2014): Zillinger (monotone shears near Couette), Wei, Zhang et al (monotone shears, shears with simple critical pts), Bedrossian, CZ, Vicol (smooth vortices), CZ, Zillinger (singular vortices).
- 2D NSE (after 2014): Wei, Zhang et al (Kolmogorov), CZ, Elgindi, Widmayer (Poiseuille).
- 2D compressible Euler/NSE (2020): Antonelli, Dolce, Marcati (Couette)
- 2D stratified fluids/Boussinesq (after 2018): Yang, Lin (inviscid Couette), Zillinger and Deng, Wu, Zhang (viscous and diffusive Couette), Bianchini, CZ, Dolce (inviscid close to Couette) Masmoudi, Said-Houari, Zhao (inviscid and diffusive Couette).

Nonlinear results

2D Euler

- Same as Bedrossian-Masmoudi for monotonic flows $\boldsymbol{U} = (u(y), 0)$ on $\mathbb{T} \times [-1, 1]$ (Ionescu, Jia '19 and Masmoudi, Zhao '19)
- Same as Bedrossian-Masmoudi for for the point-vortex (Ionescu, Jia '19)

2D NSE

- Nonlinear transition threshold: Wei, Zhang et al (Kolmogorov on rectangular torus), CZ, Elgindi, Widmayer (Poiseuille).

2D Boussinesq

- Zillinger and Deng, Wu, Zhang (viscous and diffusive Couette), Masmoudi, Said-Houari, Zhao (inviscid and diffusive Couette).

Steady states

Local structure of steady states

- Lin, Zeng '10: there are steady states near Couette in H^s ($s < 3/2$), with cat's eye structure (i.e. nontrivial x-dependence). All steady states near Couette in H^s ($s > 3/2$) are shears.
- Choffrut, Sverak '12: Neighborhoods of non-degenerate steady states in an annulus can contain only non-degenerate steady states.
- Constantin, Drivas, Ginsberg '20: there are perturbations of non-degenerate Arnold stable steady states that are non-degenerate Arnold stable

Local vs global degeneracies

Write Euler near a shear $(u(y), 0)$:

$$u\partial_x\omega - u''\Delta^{-1}\partial_x\omega + \boldsymbol{u}\cdot\nabla\omega = 0$$

- Local degeneracy: u has a (simple) critical point
- Global degeneracy: The kernel of the linear operator

$$\mathcal{L}_u = u\partial_x - u''\Delta^{-1}\partial_x$$

  is "big" (does not only contain shears)

Question: what is the role of degeneracies in the local structure of steady states?

Examples

- Couette: $v(y) = y$, on $\mathbb{T} \times [-1, 1]$, is non-degenerate
- Poiseuille: $v(y) = y^2$, on $\mathbb{T} \times [-1, 1]$, is locally degenerate but the kernel of
$$\mathcal{L}_P = y^2 \partial_x - 2\Delta^{-1} \partial_x$$
only contains shears
- Kolmogorov: $v(y) = \sin y$, on $\mathbb{T}^2$ is both locally and globally degenerate, since the kernel of
$$\mathcal{L}_K = \sin y (1 + \Delta^{-1}) \partial_x$$
contains also $\{\sin x, \cos x\}$. This does not happen on a rectangular torus $\mathbb{T}^2_\delta := [0, 2\pi\delta] \times [0, 2\pi]$, $\delta > 0$ with $\delta \notin \mathbb{N}$.

The Kolmogorov flow

Steady Euler flows

Any steady Euler flows $\boldsymbol{U} = \nabla^{\perp}\Psi$ satisfies

$$\nabla^{\perp}\Psi \cdot \nabla\Delta\Psi = 0.$$

Hence, if

$$\Delta\Psi = F(\Psi), \qquad F \in C^1,$$

then Ψ is a steady solution. Kolmogorov flow is $\boldsymbol{U}_K = (\sin y, 0)$, hence $\Psi_K := \cos(y)$, and

$$\Delta\Psi_K = F_K(\Psi_K), \qquad F_K(z) = -z.$$

Structures near Kolmogorov

Structures near Kolmogorov [CZ, Elgindi, Widmayer '20]

There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ there exist analytic functions $\Psi_\varepsilon \in C^\omega(\mathbb{T}^2)$ and $F_\varepsilon \in C^\omega(\mathbb{R})$ satisfying

$$\Delta\Psi_\varepsilon = F_\varepsilon(\Psi_\varepsilon) \tag{1}$$

and

$$\|\cos(y) - \Psi_\varepsilon\|_{C^\omega(\mathbb{T}^2)} = O(\varepsilon), \tag{2}$$

with

$$\langle \Psi_\varepsilon, \cos(x)\cos(4y)\rangle = -\varepsilon^2 \frac{\pi^2}{128} + O(\varepsilon^3). \tag{3}$$

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- F_ε is a polynomial of degree 5, so if $\Psi_\varepsilon \in H^2$ then, by elliptic regularity, it is analytic.
- There are families of non-trivial (i.e. not in the kernel of $\mathcal{L}_K$), non-shear and stationary solutions $U_\varepsilon := \nabla^\perp \Psi_\varepsilon : \mathbb{T}^2 \to \mathbb{R}^2$ of the incompressible Euler equations.

The general strategy

To find a larger class of solutions near Kolmogorov, we make the ansatz

$$\Psi_\varepsilon = \Psi_K + \varepsilon\psi, \qquad F_\varepsilon = F_K + \varepsilon f,$$

which yields a nonlinear elliptic equation for ψ, with f to be determined as well,

$$\Delta\psi + \psi = f(\Psi_K + \varepsilon\psi).$$

GOAL

Find (f, ψ), with ψ even in x and y separately, such that

$$\Delta\psi + \psi = f(\cos(y) + \varepsilon\cos(x) + \varepsilon\psi), \qquad \text{with } \psi \perp \ker(\Delta + 1),$$

with f as a quintic polynomial (with coefficients $A, B \in \mathbb{R}$ to be determined as functionals of ψ and $\varepsilon > 0$)

$$f(A, B; s) = As + Bs^3 + \frac{1}{5}s^5.$$

Explicitly

This amounts to solve

$$\begin{aligned}\Delta\psi + \psi = {} & A\cos(y) + B\cos^3(y) + \frac{1}{5}\cos^5(y) \\ & + \varepsilon\psi\Big(A + 3B\cos^2(y) + \cos^4(y)\Big) \\ & + \varepsilon\cos(x)\Big(A + 3B\cos^2(y) + \cos^4(y)\Big) \\ & + R(B,\psi,\varepsilon; x, y),\end{aligned}$$

with $R(B,\psi,\varepsilon;x,y) = O(\varepsilon^2)$.

Solvability conditions (SC)

$$\langle f(A,B;\cos(y) + \varepsilon\cos(x) + \varepsilon\psi), \cos(x)\rangle = 0$$
$$\langle f(A,B;\cos(y) + \varepsilon\cos(x) + \varepsilon\psi), \cos(y)\rangle = 0.$$

The contraction set-up

$$X := \Big\{ \psi \in H^2 : \psi(-x,y) = \psi(x,-y) = \psi(x,y), \quad \psi \perp \cos(y), \cos(x),$$
$$\big|\langle \psi, \cos^2(y)\cos(x)\rangle\big| + \big|\langle \psi, \cos^4(y)\cos(x)\rangle\big| \le \frac{1}{100}, \quad \|\psi\|_{H^2} \le 10 \Big\}.$$

The map $K_\varepsilon : X \to H^2$

We look for a fixed point of

$$\psi \mapsto \big[(x,y) \mapsto (1+\Delta)^{-1} f(A(\psi;\varepsilon), B(\psi,\varepsilon); \cos(y) + \varepsilon\cos(x) + \varepsilon\psi)\big]$$

$$\Psi_\varepsilon = \cos(y) + \varepsilon\,[\cos(x) + c_0\cos(3y) - c_1\cos(5y)]$$
$$+ \varepsilon^2 \left[-c_2\cos(x)\cos(4y) - \frac{1}{32} b_1 \cos(3y) - c_3\cos(7y) + c_4\cos(9y) \right]$$
$$+ O(\varepsilon^3).$$

Many such families $(\Psi_\varepsilon)_\varepsilon$ exist (can F_ε at order ε^2).

Remarks and consequences

Inviscid damping

$$\begin{cases} \partial_t \omega + \mathcal{L}_K \omega = -\boldsymbol{u} \cdot \nabla \omega, \\ \boldsymbol{u} = \nabla^\perp \psi, \quad \Delta \psi = \omega. \end{cases}$$

- Wei, Zhang, Zhao '17: there is linear inviscid damping, namely, linearly all modes away from the kernel of $\mathcal{L}_K$ decay.
- CZ, Elgindi, Widmayer '20: the result cannot be extended perturbatively at the nonlinear level, no matter the regularity. The dynamics near Kolmogorov on $\mathbb{T}^2$ is much richer.

Obstructions on the Square Torus

Not all directions are good! There are elements of $\ker \mathcal{L}_K$ which cannot arise as projections of stationary states.

Obstructions on the Torus

If for some $\ell \in \mathbb{N}$, $\ell \geq 2$,

$$\frac{\mathbb{P}_K(\Omega_* - \cos(y))}{\|\mathbb{P}_K(\Omega_* - \cos(y))\|_{L^2}} = \sin(\ell y) + \cos(x),$$

then there exists $\varepsilon_0 > 0$ small so that if $\|\Omega_* - \cos(y)\|_{H^6} = \varepsilon < \varepsilon_0$, then Ω_* is not a stationary solution to the $2d$ Euler equations.

Rigidity on Rectangular Tori

Rigidity near Kolmogorov on a rectangular torus

Consider the stationary solution $U_K(x, y) = (\sin(y), 0)$ on $\mathbb{T}^2_\delta$, $\delta > 0$ with $\delta \notin \mathbb{N}$. There exists $\varepsilon_0 > 0$ (depending on δ) such that if $U : \mathbb{T}^2_\delta \to \mathbb{R}^2$ is a further stationary solution to the Euler equations with

$$\|U - U_K\|_{H^3} \leq \varepsilon_0,$$

then $U = U(y)$ is necessarily a shear flow.

Rigidity near Poiseuille flow

Near Poiseuille flow, even any nearby travelling wave solution must simply be a shear flow.

Rigidity near Poiseuille

Let $s > 5$, and consider the $2d$ Euler equations on $\mathbb{T} \times [-1, 1]$

$$\partial_t U + U \cdot \nabla U + \nabla P = 0, \qquad \nabla \cdot U = 0, \qquad U_2(x, \pm 1) = 0.$$

There exists $\varepsilon_0 > 0$ such that if $U(x - ct, y)$, with $c \in \mathbb{R}$, is any traveling wave solution that satisfies

$$\|\Omega + 2y\|_{H^s} \leq \varepsilon_0, \quad \text{where } U = \nabla^\perp \Psi, \quad \Delta\Psi = \Omega,$$

then it follows that $U \equiv (U_1, 0)$, that is, U is necessarily a shear flow.

Enhanced Dissipation near Bar States on $\mathbb{T}^2$

The linearization of the Navier-Stokes equations near the bar states $\Omega_{bar} = -\mathrm{e}^{-\nu t}\cos(y)$ is then given by

$$\partial_t f + \mathrm{e}^{-\nu t}\mathcal{L}_K f = \nu \Delta f.$$

Ibrahim, Maekawa and Masmoudi '17 and Wei, Zhang, Zhao '17 showed that

$$\|\mathbb{P}_{\mathcal{D}} f(t)\|_{L^2} \lesssim \mathrm{e}^{-c_1 \nu^{1/2} t} \|\mathbb{P}_{\mathcal{D}} f(0)\|_{L^2}, \qquad \forall t \leq \frac{T}{\nu}, \qquad \mathcal{D} := (\ker \mathcal{L}_K)^{\perp}.$$

Typical nonlinear transition threshold

At the nonlinear level, there exists $\gamma \geq 0$ such that if

$$\|\mathbb{P}_{\mathcal{D}} \omega^{in}\|_X \lesssim \nu^{\gamma} \qquad \Rightarrow \qquad \|\mathbb{P}_{\mathcal{D}} \omega(t)\|_{L^2} \lesssim \mathrm{e}^{-c_1 \nu^{1/2} t} \left\|\mathbb{P}_{\mathcal{D}} \omega^{in}\right\|_{L^2}$$

- True for rectangular tori (Wei, Zhang, Zhao '17)
- True for Poiseuille flow (CZ, Elgindi, Widmayer '19)

No Threshold near Bar States on $\mathbb{T}^2$

No nonlinear threshold

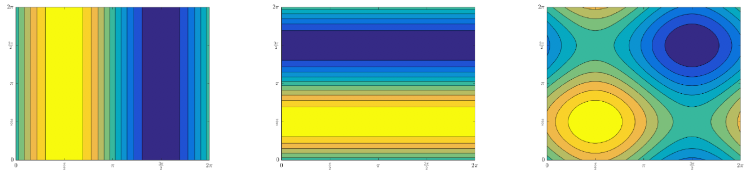
For any $\nu > 0$ there exists $0 < \varepsilon_0 \ll \nu$ with the following property: let $0 < \varepsilon \leq \varepsilon_0$ and let $\Omega_\varepsilon = \Delta\Psi_\varepsilon$ be the vorticity of the stationary Euler flow found before. Then $\mathbb{P}_{\mathcal{D}}\Omega_\varepsilon$ is not dissipated at an enhanced rate: i.e. the solution Ω^ν of the initial value problem

$$\begin{cases} \partial_t \Omega^\nu + U^\nu \cdot \nabla \Omega^\nu = \nu \Delta \Omega^\nu, \\ \Omega^\nu(0) = \Omega_\varepsilon, \end{cases}$$

on $\mathbb{T}^2$ satisfies for all $t \in [\frac{1}{2\nu}, \frac{1}{\nu}]$ the lower bound

$$\left\| \mathbb{P}_{\mathcal{D}} \Omega^\nu(t) \right\|_{L^2} \gtrsim \left\| \mathbb{P}_{\mathcal{D}} \Omega_\varepsilon \right\|_{L^2}.$$

In the right direction?



(a) x-bar: $\omega_{xbar} = \sin(x)$ (b) y-bar: $\omega_{ybar} = \sin(y)$ (c) Dipole: $\omega_{dipole} = \sin(x) + \sin(y)$

Figure 2: M. Beck, E. Cooper, G. Lord, K. Spiliopoulos

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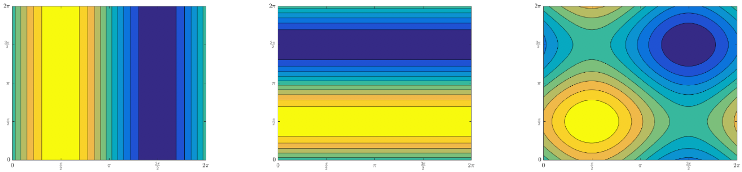

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THANK YOU